# Tree-pair Diagram Representatives, a Normal Form, and Estimating the Metric for generalizations of Thompson's Group

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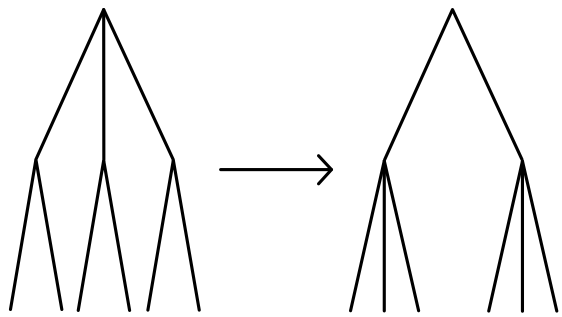
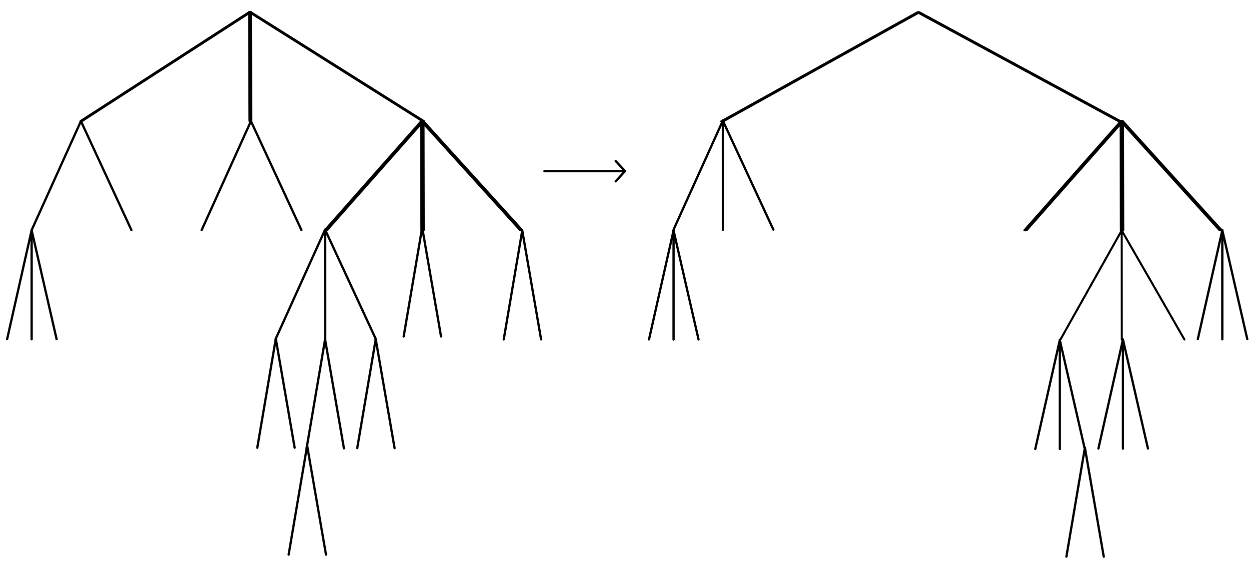
## Representatives of the identity

Tree-pair diagrams representing the identity in $F(n)$ will always consist of two identical trees. This is not the case in $F(n, m)$.





## Minimal tree-pair diagram representatives may not be unique

Minimal tree-pair diagram representatives of $F(n)$ are unique. This is not the case in $F(n,m)$.

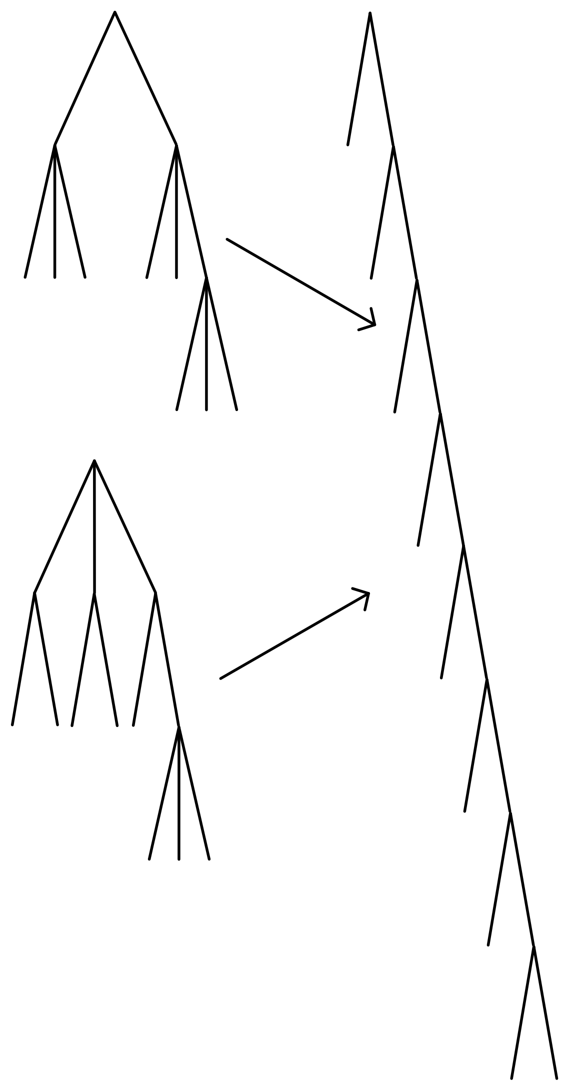


Figure 1: Two equivalent but distinct minimal tree-pair diagrams representing an element of $F(2,3)$

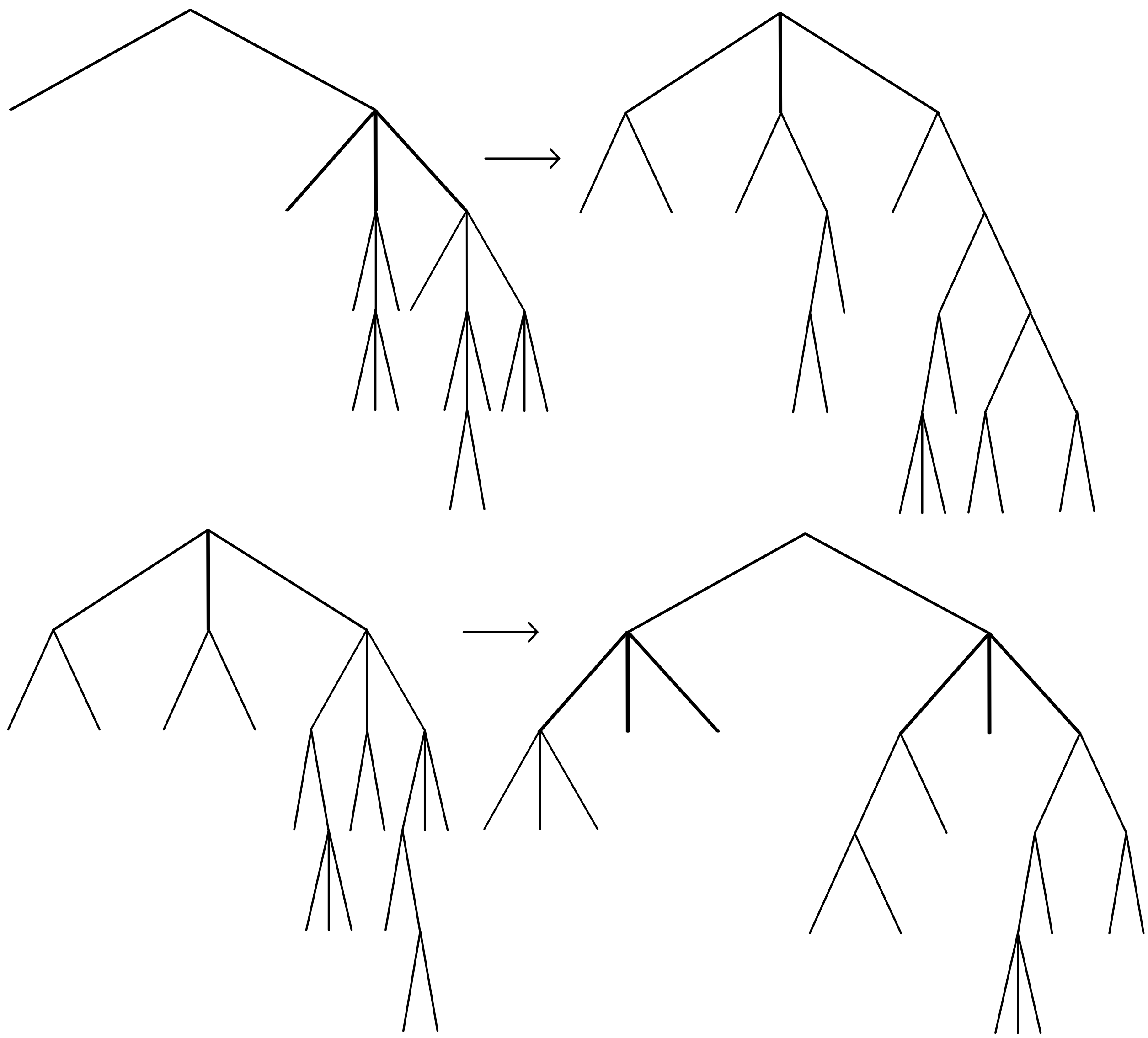


Figure 2: Two equivalent but distinct minimal tree-pair diagrams representing another element of $F(2,3)$

## To get to the minimal tree-pair diagram, we may have to add carets

Minimal tree-pair diagram representatives of $F(n)$ can always be obtained solely by caret removal. Whereas in $F(n, m)$, we may even need to add carets in order to obtain a minimal tree-pair diagram from a given tree-pair diagram.

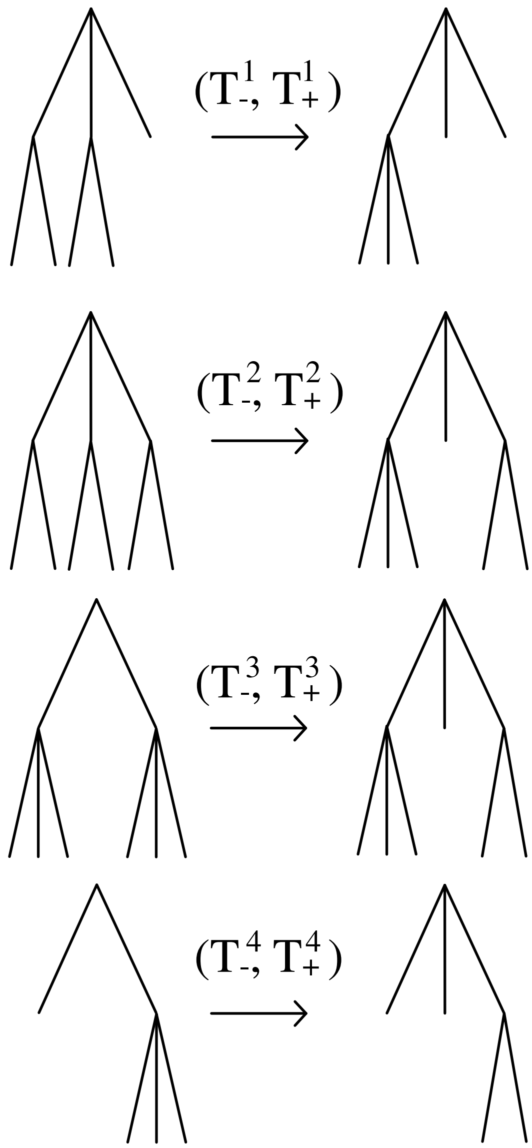



Figure 3: $(T_-^1, T_+^1)$ is a (2,3)–ary tree-pair diagram which must have carets added to it in order to be transformed into its equivalent minimal tree-pair diagram $(T_-^4, T_+^4)$

## Multiplying tree-pair diagrams in $F(n)$

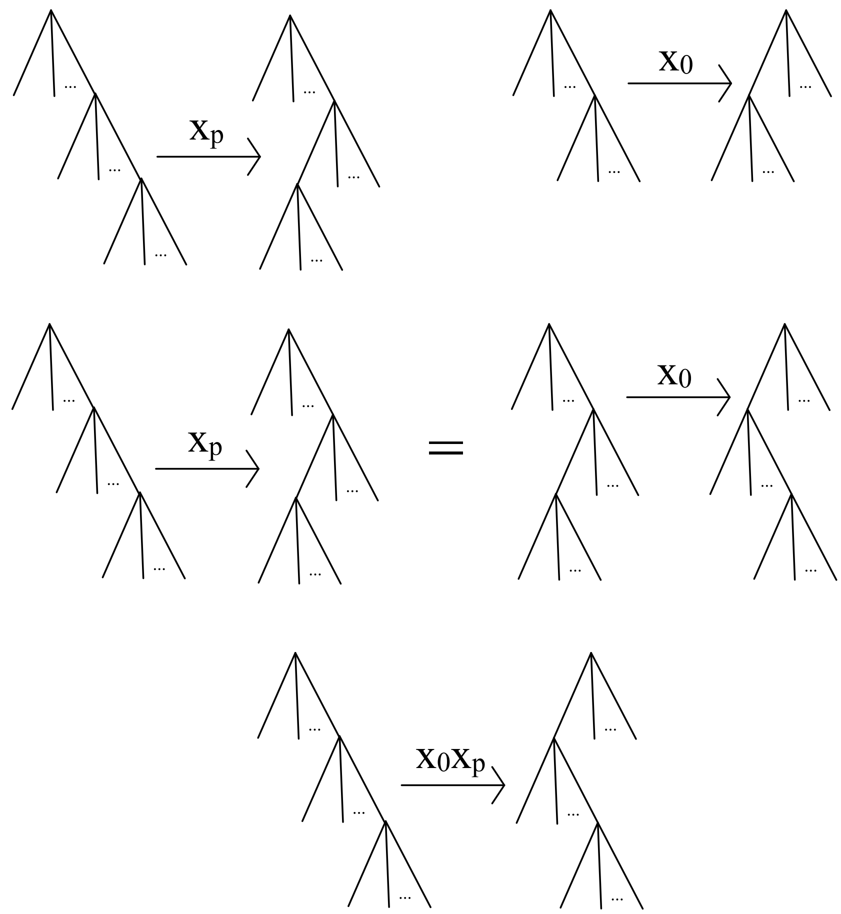



Figure 4: Composition of two elements of $F(n)$

## Multiplying tree-pair diagrams on $F(n, m)$

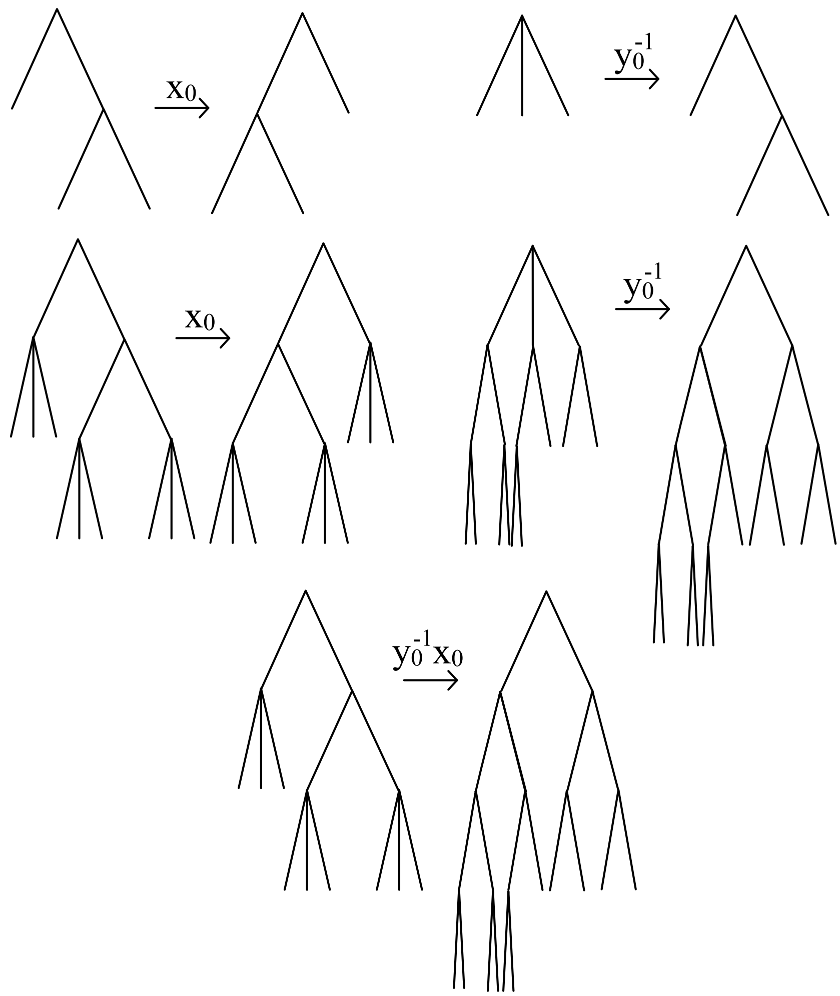
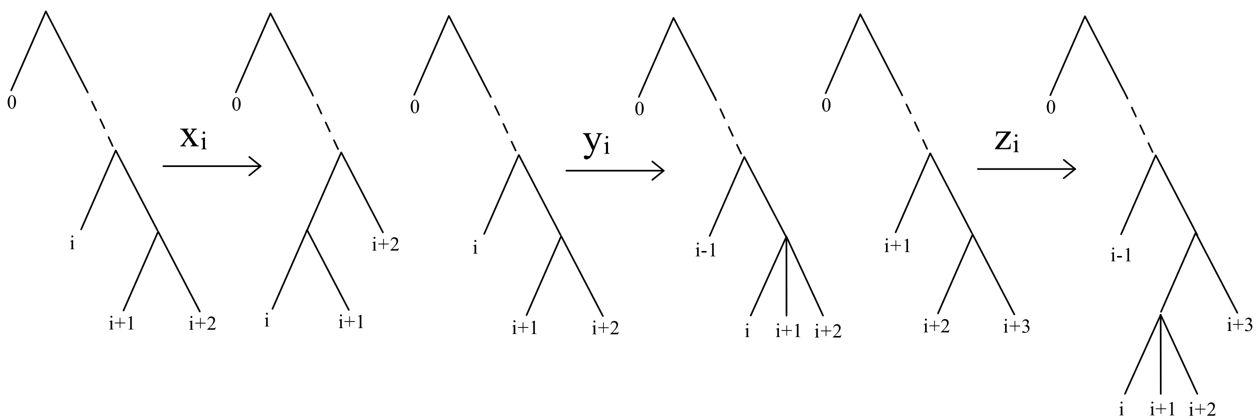



Figure 5: Composition of two elements of $F(2, 3)$

**Standard infinite presentation (Stein)**

Generators:

$$\{x_0, x_1, ..., y_0, y_1, ..., z_0, z_1, ...\}$$




Relators:

1. $x_j x_i = x_i x_{j+1}$
2. $y_j x_i = x_i y_{j+1}$
3. $z_j x_i = x_i z_{j+1}$
4. $x_j z_i = z_i x_{j+2}$
5. $y_j z_i = z_i y_{j+2}$
6. $z_j z_i = z_i z_{j+2}$

for $i < j$ and

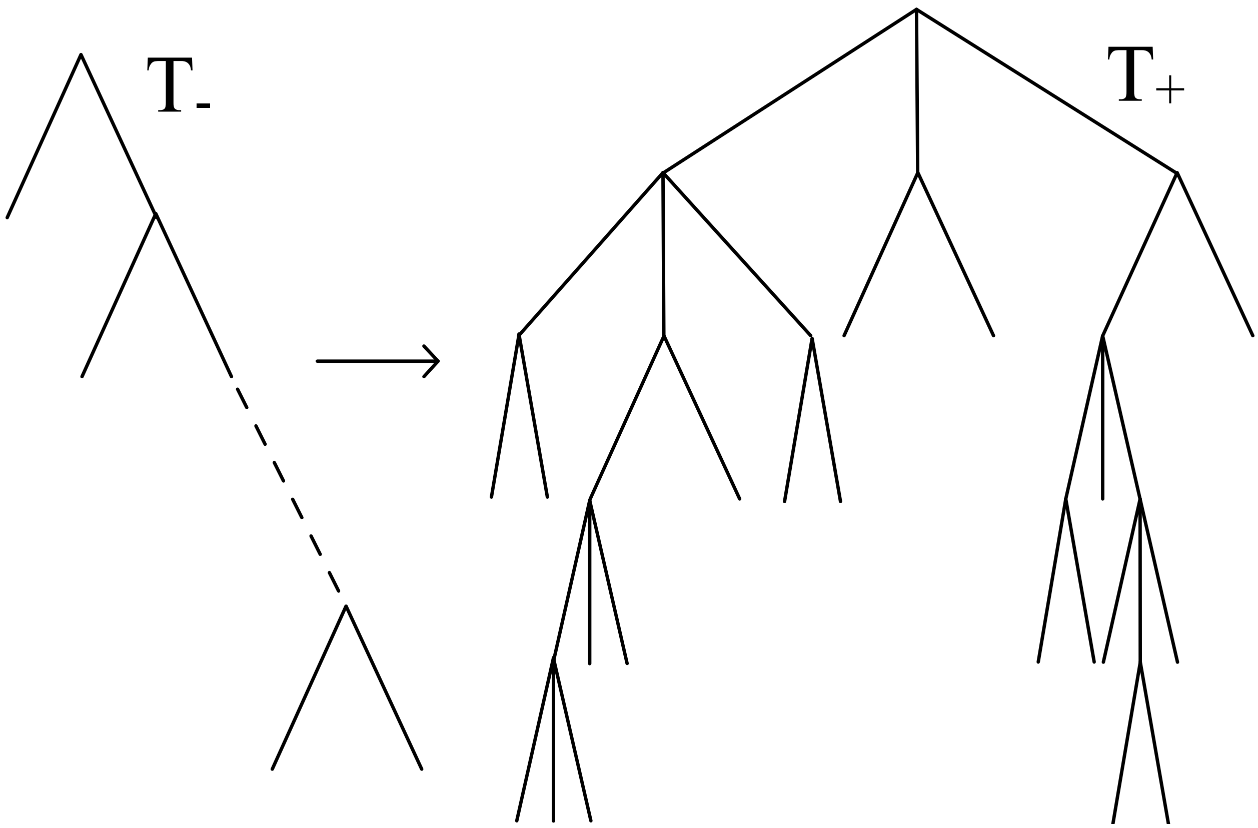
1. $y_{i+1} z_i = y_i x_{i+1} x_i$
2. $x_i z_{i+1} z_i = z_i x_{i+2} x_{i+1} x_i$

for all $i$.

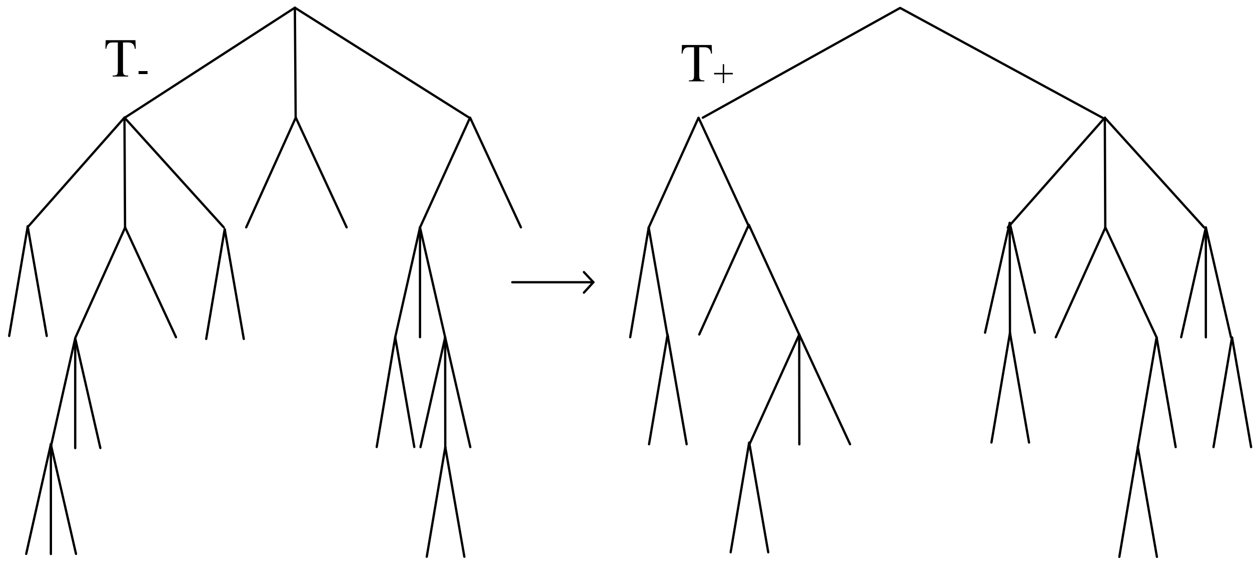
## Normal Form

For positive $w$:

$$NF(w) = \mathcal{B}_0 \cdots \mathcal{B}_k$$



## Normal Form Example



## Normal Form

The normal form of an element $w$ in $F(2,3)$ can be written as:

$$NF(w) = NF(w_+)NF(w_-^{-1})$$

. For a positive element $w$ of $F(2,3)$ the normal form is:

$$NF(w) = \mathcal{B}_0 \cdots \mathcal{B}_k$$

where

$$\mathcal{B}_0 = y_{\beta_1} \cdots y_{\beta_N}$$

such that $\beta_{i+1} > \beta_i + 1$ for all $i \in \{1, ..., N\}$ and where

$$\mathcal{B}_i = (\gamma_{\alpha i,1}^{i,1})^{e_{i,1}} \cdots (\gamma_{\alpha i,s_i}^{i,s_i})^{e_{i,s_i}}$$

such that

$$\alpha_{1,j} \neq 0 \text{ for all } j \in \{1, ..., s_1\}$$

and

$$\alpha_{i,1} = 0 \text{ and } e_{i,1} = 1 \text{ for all } i \in \{2, ..., k\}$$

and

$$\alpha_{i,j} \neq 0 \text{ for all } j \in \{2, ..., s_i\} \text{ and } i \in \{2, ..., k\}$$

and where for all $i \in \{2, ..., k\}$

$$\alpha_{i,j} \leq \sum_{l=1}^{j-1} e_{i,l} \cdot \left(\omega\left(\gamma_{\alpha_{i,l}}^{i,l}\right) - 1\right)$$

for all $j \in \{1, ..., s_i\}$.

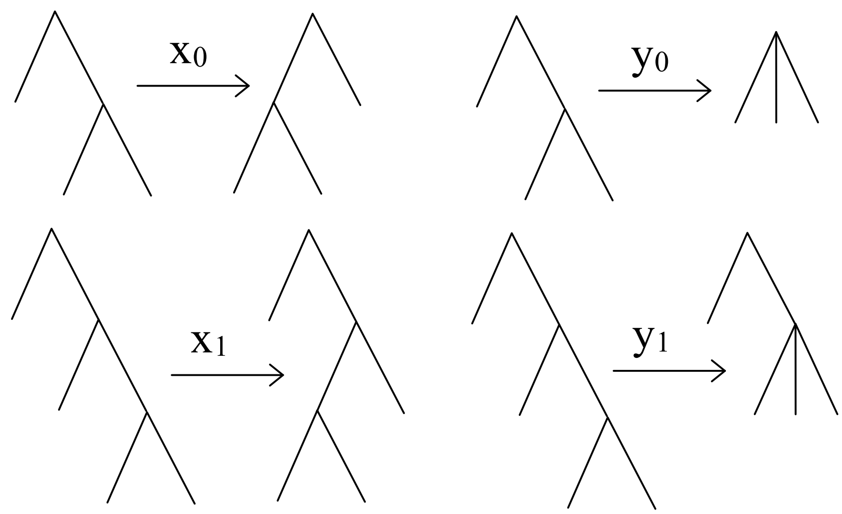
where $\omega(\gamma_i)$ is defined such that

$$\omega(\gamma_i) = \begin{cases} 2 \text{ if } \gamma_i = x_i \\ 3 \text{ if } \gamma_i = z_i \end{cases}$$

**Standard finite presentation (Stein)**

Generators:

$$\{x_0, x_1, y_0, y_1\}$$




Relators:

1. $x_2x_0 = x_0x_3$
2. $x_3x_1 = x_1x_4$
3. $y_2x_0 = x_0y_3$
4. $y_3x_1 = x_1y_4$
5. $x_1z_0 = z_0x_3$
6. $x_2z_1 = z_1x_4$
7. $y_1z_0 = z_0y_3$
8. $y_2z_1 = z_1y_4$
9. $x_0z_1z_0 = z_0x_2x_1x_0$
10. $x_1z_2z_1 = z_1x_3x_2x_1$

where $x_3 = x_1^{-1}x_2x_1$, $x_4 = x_2^{-1}x_3x_2$, $y_3 = x_1^{-1}y_2x_1$, $y_4 = x_2^{-1}y_3x_2$, $z_0 = y_1^{-1}y_0x_1x_0$, $z_1 = y_2^{-1}y_1x_2x_1$, and $z_2 = y_3^{-1}y_2x_3x_2$.

## The Metric on $F(2,3)$

The metric on $F(2,3)$ is not quasi-isometric to the number of carets or leaves in the minimal tree-pair diagram representatives of elements of $F(2,3)$ (as is the case in $F(n)$).

$$log L(w) \leq |w|_{\{x_0,x_1,y_0,y_1\}} \leq cL(w)$$

for fixed $c \in \mathbb{Z}^+$ (where $L(w)$ denotes the number of leaves in the minimal tree-pair diagram representative of $w$).

The order of the upper and lower bounds on this metric are sharp.